JOURNAL OF COMPUTATIONAL PHYSICS 21, 1–19 (1976)

# Successive Peripheral Over-Relaxation and Other Block Methods

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Received April 25, 1975; revised October 7, 1975

Successive 1-line and 2-line over-relaxation techniques for solving self-adjoint second-order elliptic partial-differential equations in a rectangle subject to a periodicity condition in one coordinate direction are shown to be equivalent to successive peripheral techniques in plane regions with holes. Algorithms are given for solving the resulting sets of equations and the numerical results obtained substantiate the equivalence. An intuitive comparison is also made between successive peripheral over-relaxation and the more usual 1-line and 2-line blocks for the model Dirichlet problem. It is concluded that the 1-peripheral block (SPOR) corresponds more to the 2-line method (S2LOR) rather than to the single line grouping (SLOR).

## INTRODUCTION

In a previous paper [2], the authors introduced a new ordering of the mesh points on a two-dimensional grid viz. around successive peripherals of the region of integration. The technique, when employed in a block over-relaxation scheme, was called successive peripheral over-relaxation (SPOR).

The purpose of this paper is to introduce new algorithms for solving the sets of equations arising from the method and also to show that for a certain class of problems, a theory does exist for SPOR. Numerical results to substantiate the theory are also given.

## 1. FORMULATION OF THE PROBLEM

Consider the solution of the self-adjoint, second-order, elliptic partial-differential equation

$$-[A(x, y)\phi_x(x, y)]_x - [C(x, y)\,\phi_y(x, y)]_y + F(x, y)\,\phi(x, y) = G(x, y), \quad (1.1)$$

for $(x, y) \in R$, where $R$ is the rectangle

$$0 \leqslant x \leqslant l_1, \qquad 0 < y < l_2,$$

with the condition that $\phi$ is periodic in the $x$-direction. The boundary conditions are

$$\phi(x, y) = \phi(x \pm l_1, y), \tag{1.2}$$

for $(x, y) \in \bar{R}$, where $\bar{R}$ is the region $0 \leqslant x \leqslant l_1$, $0 \leqslant y \leqslant l_2$, and

$$\phi(x, y) \text{ given for all } 0 \leqslant x \leqslant l_1,\ y = 0, l_2 \tag{1.3}$$

Also, $A$, $C$, $F$, $G$ are continuous in $\bar{R}$ and satisfy

$$\begin{aligned} A(x, y) &> 0, \qquad & A(x, y) &= A(x \pm l_1, y), \\ C(x, y) &> 0, \qquad & C(x, y) &= C(x \pm l_1, y), \\ F(x, y) &\geqslant 0, \qquad & F(x, y) &= F(x \pm l_1, y), \\ & & G(x, y) &= G(x \pm l_1, y), \end{aligned} \tag{1.4}$$

for $(x, y) \in \bar{R}$.

This problem has been treated by Wood [7], but without much consideration of the computational aspects involved. In this paper, use will be made of some of the theoretical results given by Wood as well as the derivation of efficient computational methods for solving the problem. The problem itself can be thought of as solving (1.1) subject to the conditions (1.2)–(1.4) over an infinite strip with a periodically occurring pattern or over the surface of a cylinder with the function values being given on the boundaries. From this latter point of view it is a Dirichlet problem and topologically equivalent to a plane region with a hole in it.

## 2. The Difference Equations

If mesh lines parallel to the coordinate axes are superimposed on the region $R$, then for any mesh point $(x, y)$

$$x = ih, \qquad i = 0, 1, \ldots, n-1,$$

$$y = jk, \qquad j = 1, 2, \ldots, m,$$

where $nh = l_1$, $(m+1)k = l_2$, and the periodic boundary condition is obtained by considering the integer suffix $i$ as interpreted modulo $n$.

The derivatives in (1.1) are replaced by the equivalent weighted difference representations of the form

$$[A(x, y)\ \phi_x(x, y)]_x = \{A(x + (h/2), y)[\phi(x + h, y) - \phi(x, y)] \\ - A(x - (h/2), y)[\phi(x, y) - \phi(x - h, y)]\}/h^2, \quad (2.1)$$

or, denoting $\phi(x, y) = \phi(ih, jk)$ by $\phi_{ij}$, etc.,

$$[A(x, y)\ \phi_x(x, y)]_x = [A_{i+(1/2),j}(\phi_{i+1,j} - \phi_{i,j}) - A_{i-(1/2),j}(\phi_{i,j} - \phi_{i-1,j})]/h^2, \quad (2.2)$$

the right-hand side being an approximation to the derivative on the left-hand side, evaluated at the point $(x, y)$. A similar expression holds for $[C(x, y)\ \phi_y(x, y)]_y$ at the point $(x, y)$. Substitution of the approximation (2.2) together with that for $[C(x, y)\ \phi_y(x, y)]_y$ into (1.1) yields a finite-difference representation of (1.1) at the point $(ih, jk)$ in the form

$$v_{i,j}\phi_{i,j-1} + a_{i,j}\phi_{i-1,j} + b_{i,j}\phi_{i,j} + c_{i,j}\phi_{i+1,j} + u_{i,j}\phi_{i,j+1} = s_{i,j} + t_{i,j},$$

where

$$\begin{aligned} a_{i,j} &= -k^2 A_{i-(1/2),j}, \qquad c_{i,j} = -k^2 A_{i+(1/2),j}, \\ v_{i,j} &= -h^2 C_{i,j-(1/2)}, \qquad u_{i,j} = -h^2 C_{i,j+(1/2)}, \\ b_{i,j} &= h^2k^2 F_{i,j} - a_{i,j} - c_{i,j} - u_{i,j} - v_{i,j}, \\ s_{i,j} &= h^2k^2 G_{i,j}, \end{aligned} \quad (2.4)$$

and $t_{i,j}$ represents the truncation error term.

The solution of (1.1) subject to the specified conditions is then approximated by the solution of the difference equation (2.3) with the truncation error term neglected. This set of equations may be written in matrix notation as

$$A\boldsymbol{\phi} = \mathbf{s}. \quad (2.5)$$

If the points are ordered along the $x$-lines, then $A$ is partitioned into blocks corresponding to the lines $y = jk$ and takes the form

$$A = \begin{bmatrix} B_1 & C_1 & & & & \\ A_2 & B_2 & C_2 & & \mathbf{0} & \\ & A_3 & B_3 & C_3 & & \\ \mathbf{0} & & \ddots & \ddots & \ddots & \\ & & & A_{m-1} & B_{m-1} & C_{m-1} \\ & & & & A_m & B_m \end{bmatrix}, \quad (2.6)$$

where $A_j$, $B_j$, $C_j$ are square matrices of order $n$ such that

$$B_j = \begin{bmatrix} b_{0,j} & c_{0,j} & & & a_{0,j} \\ a_{1,j} & b_{1,j} & c_{1,j} & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & a_{n-2,j} & b_{n-2,j} & c_{n-2,j} \\ c_{n-1,j} & & & a_{n-1,j} & b_{n-1,j} \end{bmatrix}, \tag{2.7}$$

and

$$\begin{aligned} C_j &= \operatorname{diag}(u_{0,j}, u_{1,j}, \dots, u_{n-1,j}), \\ A_j &= \operatorname{diag}(v_{0,j}, v_{1,j}, \dots, v_{n-1,j}), \end{aligned} \tag{2.8}$$

for $j = 1, 2, \dots, m$. The vectors $\boldsymbol{\phi}$ and $\mathbf{s}$ of (2.5) are then partitioned relative to the matrix $A$ of (2.6). Iterative methods for the efficient solution of (2.5) will now be considered.

## 3. Successive Over-Relaxation Techniques

Consider a natural ordering of the lines $y = jk$. If the coefficient matrix $A$ of (2.5) is partitioned so that its diagonal submatrices are all of order $(1 \times 1)$, i.e., the point case then the matrix properties depend on whether $n$, the number of points on each line is even or odd. If $n$ is odd, then the matrix does not possess Property $A$, and the successive over-relaxation theory is not valid. For $n$ even, however, the matrix does possess Property $A$ and though a natural ordering of the points on each line does not lead to a consistently ordered set of equations, a suitable ordering can be found to enable the theory to be applicable.

In this case, the usual relationship, viz.

$$\frac{\lambda + \omega - 1}{\omega} = \lambda^{1/2}\mu, \tag{3.1}$$

exists between the eigenvalues $\lambda$, $\mu$ of the SOR and associated Jacobi matrices, respectively, and the over-relaxation factor $\omega$, the optimum value of which is given by,

$$\bar{\omega} = 2/(1 + (1 - \bar{\mu}^2)^{1/2}) \tag{3.2}$$

where $\bar{\mu}$ is the spectral radius of the Jacobi matrix.

If (1.1) reduces to Laplace's equation subject to the boundary conditions (1.2) and (1.3), the finite-difference analog at the point $(i, j)$ can be written as,

$$\phi_{ij} = \theta_x[\phi_{i-1,j} + \phi_{i+1,j}] + \theta_y[\phi_{i,j-1} + \phi_{i,j+1}], \tag{3.3}$$

where

$$\theta_x = \frac{k^2}{2(h^2 + k^2)}; \qquad \theta_y = \frac{h^2}{2(h^2 + k^2)}.$$

It then can be shown [7] that in the case when the SOR theory applies, the spectral radius of the point SOR matrix is given asymptotically by,

$$\bar{\lambda} \simeq 1 - 2(2\theta_y)^{1/2}(\pi/(m + 1)), \tag{3.4}$$

for large values of $m$, the number of lines of unknowns. Thus, for a square mesh, the asymptotic rate of convergence of the point SOR matrix satisfies,

$$R(\text{SOR}) \simeq 2^{1/2}(\pi/(m + 1)),$$

and consequently, is dependent only on $m$, the number of lines.

For problems of any computational difficulty, however, point methods are unlikely to be used, and where possible, block methods would be employed. If then $A$ is partitioned so that each diagonal submatrix consists of mesh points on successive groups of $p$-lines in the $x$ direction, then the matrix is block 2 cyclic and a natural ordering of the lines is consistent. The block SOR theory is then immediately valid so that (3.1) and (3.2) again hold, but now, $\lambda$ is an eigenvalue of the $p$-line SOR matrix etc. For Laplace's equation, the diagonal submatrices $B_j$ of (2.7) are then given by,

$$B_j = \begin{bmatrix} 1 & -\theta_x & & & -\theta_x \\ -\theta_x & 1 & -\theta_x & & \\ & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & & -\theta_x & 1 & -\theta_x \\ -\theta_x & & & -\theta_x & 1 \end{bmatrix}, \tag{3.5}$$

with

$$A_j = C_j = -\theta_y I. \tag{3.6}$$

For $p = 1$, the eigenvalues of the 1-line block Jacobi matrix $D^{-1}(L + U)$, where $A = D - L - U$, are then

$$\mu_{r,s} = \frac{2\theta_y \cos(s\pi/(m + 1))}{1 - 2\theta_x \cos(2\pi r/n)}, \qquad s = 1, 2, \ldots, m, \quad r = 0, 1, \ldots, n - 1, \tag{3.7}$$

so that the spectral radius is

$$\bar{\mu} = \cos(\pi/(m + 1)). \tag{3.8}$$

Consequently, the optimum value of $\omega$, the spectral radius of the one-line SOR matrix, and the corresponding rate of convergence are all dependent only on $m$, the number of lines.

The successive line over-relaxation method (SLOR) for solving (2.5) with $A$ as given by (2.6) is defined by

$$B_j\boldsymbol{\phi}_j^{(r+1)} = B_j\boldsymbol{\phi}_j^{(r)} + \omega\{\mathbf{s}_j - A_j\boldsymbol{\phi}_{j-1}^{(r+1)} - C_j\boldsymbol{\phi}_{j+1}^{(r)} - B_j\boldsymbol{\phi}_j^{(r)}\}. \tag{3.9}$$

(This form, because of the periodicity condition, corresponds very closely to the SPOR method for the Dirichlet problem [2]). The computational procedure involves repeatedly solving sets of equations of the form

$$B_j\boldsymbol{\phi}_j = \mathbf{z}_j\,, \tag{3.10}$$

where $B_j$ is as given in (2.7). To solve such systems, efficient algorithms have been given previously by Atkinson and Evans [1].

If (1.1) is self-adjoint, then $A(x, y) = A(x)$ and $C(x, y) = C(y)$ and the coefficient matrix $A$ of (2.5) is symmetric and positive definite. Under these conditions, a normalized form of the iteration (3.9) is defined by the following equations [5].

$$\begin{aligned}(T_j'T_j)\,\tilde{\mathbf{v}}_j^{(r+1)} &= [\tilde{\mathbf{s}}_j - \tilde{A}_j\mathbf{v}_{j-1}^{(r)} - \tilde{C}_j\mathbf{v}_{j+1}^{(r)}],\\ \mathbf{v}_j^{(r+1)} &= \omega[\tilde{\mathbf{v}}_j^{(r+1)} - \mathbf{v}_j^{(r)}] + \mathbf{v}_j^{(r)},\end{aligned} \tag{3.11}$$

for $1 \leqslant j \leqslant m$, where $B_j$ has been expressed as

$$B_j = D_jT_j'T_jD_j\,, \tag{3.12}$$

and where the prime denotes transpose. Here, $D_j$ is a positive diagonal matrix' $T_j$ is an upper triangular matrix and

$$\begin{aligned}&D_j\boldsymbol{\phi}_j = \mathbf{v}_j\,; \qquad D_j\mathbf{s}_j = \tilde{\mathbf{s}}_j\,, \qquad \text{for} \quad 1 \leqslant j \leqslant m,\\ &\tilde{A}_j = D_j^{-1}A_jD_{j-1}^{-1}\,, \qquad \text{for} \quad 2 \leqslant j \leqslant m,\\ &\tilde{C}_j = D_j^{-1}C_jD_{j+1}^{-1}\,, \qquad \text{for} \quad 1 \leqslant j \leqslant m-1.\end{aligned} \tag{3.13}$$

If now the matrix $A$ of (2.5) is partitioned so that each diagonal submatrix consists of points ordered alternately across each pair of successive lines (i.e., $p = 2$) then Eq. (3.9) represents a method of solution corresponding to that of 2-line successive over-relaxation (S2LOR), but once again because of the periodicity condition, represents closely the 2-peripheral method (S2POR) on a plane region with a hole in it [2]. Now, systems of equations of the form

$$B_q\boldsymbol{\phi}_q = \mathbf{z}_q\,, \tag{3.14}$$

must be solved repeatedly, where $B_q$ is a square matrix of order $2n$ of the form

$$\mathbf{B}_q = \begin{bmatrix} b_{0,j} & u_{0,j} & c_{0,j} & & & a_{0,j} & 0 \\ v_{0,j+1} & b_{0,j+1} & 0 & c_{0,j+1} & & & a_{0,j+1} \\ a_{1,j} & 0 & b_{1,j} & u_{1,j} & c_{1,j} & & \\ & a_{1,j+1} & v_{1,j+1} & b_{1,j+1} & 0 & & \mathbf{0} \\ & & \ddots & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & & & & \\ c_{n-1,j} & & & a_{n-1,j} & 0 & b_{n-1,j} & u_{n-1,j} \\ 0 & c_{n-1,j+1} & & & a_{n-1,j} & v_{n-1,j+1} & b_{n-1,j+1} \end{bmatrix}$$

where $j = (2q - 1)$ for $q = 1, 2, \ldots, m/2$.

This solution may be effected computationally using an algorithm given by Benson and Evans [3], or using the Choleski factorization as given below.

The $B_q$ of (3.15) is from the more general set of matrices of the form

$$M = \begin{bmatrix} c_1 & -d_1 & -e_1 & & & -a_1 & -b_1 \\ -b_2 & c_2 & -d_2 & -e_2 & & & -a_2 \\ -a_3 & -b_3 & c_3 & -d_3 & -e_3 & & \\ & -a_4 & -b_4 & c_4 & -d_4 & -e_4 & \mathbf{0} \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ \mathbf{0} & & & & & & \\ & & -a_{N-2} & -b_{N-2} & c_{N-2} & -d_{N-2} & -e_{N-2} \\ -e_{N-1} & & & -a_{N-1} & -b_{N-1} & c_{N-1} & -d_{N-1} \\ -d_N & -e_N & & & -a_N & -b_N & c_N \end{bmatrix}. \qquad (3.16)$$

Formally expressing $M$ in the form $M = LU$, where

$$L = \begin{bmatrix} \omega_1 & & & & & & \\ \beta_1 & \omega_2 & & & & & \\ \alpha_1 & \beta_2 & \omega_3 & & \mathbf{0} & & \\ \ddots & \ddots & \ddots & & & & \\ & \mathbf{0} & & & & & \\ & \alpha_{N-4} & \beta_{N-3} & \omega_{N-2} & & & \\ \gamma_1 & \gamma_2 \cdots \gamma_{N-4} & (\gamma_{N-3} + \alpha_{N-3}) & (\beta_{N-2} + \gamma_{N-2}) & \omega_{N-1} & & \\ \delta_1 & \delta_2 \cdots \delta_{N-4} & \delta_{N-3} & (\alpha_{N-2} + \delta_{N-2}) & X & \omega_N & \end{bmatrix}, \qquad (3.17)$$

$$U = \begin{bmatrix} 1 & f_1 & g_1 & & & \mathbf{0} & h_1 & & m_1 \\ & 1 & f_2 & g_2 & & & h_2 & & m_2 \\ & & \ddots & \ddots & \ddots & & | & & | \\ & & & 1 & f_{N-4} & g_{N-4} & h_{N-4} & & m_{N-4} \\ & & & & 1 & f_{N-3} & (g_{N-3} + h_{N-3}) & & m_{N-3} \\ & \mathbf{0} & & & & 1 & (f_{N-2} + h_{N-2}) & & (g_{N-2} + m_{N-2}) \\ & & & & & & 1 & & Y \\ & & & & & & & & 1 \end{bmatrix}, \tag{3.18}$$

forming the product $LU$, and equating elements with the corresponding coefficients of $A$, yields the following equations for the elements of $L$ and $U$.

$$\begin{gathered} \omega_1 = c_1\,; \quad \beta_1 = -b_2\,; \quad \alpha_1 = -a_3\,; \quad \gamma_1 = -e_{N-1}\,; \quad \delta_1 = -d_N\,; \\ f_1 = -d_1/\omega_1\,; \quad g_1 = -e_1/\omega_1\,; \quad h_1 = -a_1/\omega_1\,; \quad m_1 = -b_1/\omega_1\,. \end{gathered} \tag{3.19}$$

$$\begin{aligned} &\omega_2 = c_2 - \beta_1 f_1\,; \quad \beta_2 = -(b_3 + \alpha_1 f_1); \quad \alpha_2 = -a_4\,; \quad \gamma_2 = -\gamma_1 f_1\,; \\ &\delta_2 = -(e_N + \delta_1 f_1); \quad f_2 = -(d_2 + \beta_1 g_1)/\omega_2; \quad g_2 = -e_2/\omega_2\,; \\ &h_2 = -\beta_1 h_1/\omega_2\,; \quad m_2 = -(a_2 + \beta_1 m_1)/\omega_2\,, \end{aligned} \tag{3.20}$$

and for $i = 3, 4,\ldots, N-2$

$$\begin{gathered} \omega_i = c_i - \beta_{i-1} f_{i-1} - \alpha_{i-2} g_{i-2}\,; \\ \beta_i = -(b_{i+1} + \alpha_{i-1} f_{i-1}); \qquad \alpha_i = -a_{i+2}\,; \\ \gamma_i = -(\gamma_{i-1} f_{i-1} + \gamma_{i-2} g_{i-2}); \qquad \delta_i = -(\delta_{i-1} f_{i-1} + \delta_{i-2} g_{i-2}); \\ f_i = -(d_i + \beta_{i-1} g_{i-1})/\omega_i; \qquad g_i - e_i/\omega_i\,; \\ h_i = -(\beta_{i-1} h_{i-1} + \alpha_{i-2} h_{i-2})/\omega_i; \qquad m_i = -(\beta_{i-1} m_{i-1} + \alpha_{i-2} m_{i-2})/\omega_i\,. \end{gathered} \tag{3.21}$$

Finally,

$$\begin{aligned} \omega_{N-1} = c_{N-1} &- (\beta_{N-2} + \gamma_{N-2})(f_{N-2} + h_{N-2}) \\ &- (\gamma_{N-3} + \alpha_{N-3})(g_{N-3} + h_{N-3}) - \sum_{k=1}^{N-4} \gamma_k h_k\,, \end{aligned}$$

$$X = -\Big(b_N + (\alpha_{N-2} + \delta_{N-2})(f_{N-2} + h_{N-2}) + \delta_{N-3}(g_{N-3} + h_{N-3}) + \sum_{k=1}^{N-4} \delta_k h_k\Big), \tag{3.22}$$

$$Y = (-1/\omega_{N-1}) \Big[ d_{N-1} + (\gamma_{N-2} + \beta_{N-2})(g_{N-2} + m_{N-2}) + (\gamma_{N-3} + \alpha_{N-3})\, m_{N-3} + \sum_{k=1}^{N-4} \gamma_k m_k \Big],$$

$$\omega_N = c_N - XY - (\delta_{N-2} + \alpha_{N-2})(g_{N-2} + m_{N-2}) - \sum_{k=1}^{N-3} \delta_k m_k \,. \tag{3.23}$$

Having computed these coefficients, the system

$$M\boldsymbol{\phi} = \mathbf{k}, \qquad \text{i.e.,} \qquad LU\boldsymbol{\phi} = \mathbf{k},$$

may be solved in the usual manner by putting

$$U\boldsymbol{\phi} = \mathbf{y}, \qquad L\mathbf{y} = \mathbf{k}.$$

Thus,

$$y_1 = k_1/\omega_1\,; \qquad y_2 = (1/\omega_2)(k_2 - \beta_1 y_1),$$

and for $i = 3, 4,\ldots, N-2$,

$$y_i = (1/\omega_i)(k_i - \beta_{i-1} y_{i-1} - \alpha_{i-2} y_{i-2}).$$

Finally,

$$y_{N-1} = (1/\omega_{N-1}) \Big( k_{N-1} - (\beta_{N-2} + \gamma_{N-2})\, y_{N-2} - (\alpha_{N-3} + \gamma_{N-3})\, y_{N-3} - \sum_{k=1}^{N-4} \gamma_k y_k \Big), \tag{3.24}$$

and

$$y_N = (1/\omega_N) \Big( k_N - X y_{N-1} - (\alpha_{N-2} + \delta_{N-2})\, y_{N-2} - \sum_{k=1}^{N-3} \delta_k y_k \Big).$$

Then for the solution

$$\phi_N = y_N\,, \qquad \phi_{N-1} = y_{N-1} - Y y_N\,, \tag{3.25}$$

and for $i = N-2, N-1,\ldots, 2, 1$

$$\phi_i = y_i - f_i \phi_{i+1} - g_i \phi_{i+2} - h_i \phi_{N-1} - m_i \phi_N \,.$$

However, if the $B_q$ in (3.14) is symmetric, a more efficient form of the algorithm can be developed in normalized form following Cuthill and Varga [4].

If in the system of equations

$$M\boldsymbol{\phi} = \mathbf{k}$$

the matrix $M$ is symmetric, positive definite and of the form

$$\begin{bmatrix} c_1 & b_1 & a_1 & & & a_{N-1} & b_N \\ b_1 & c_2 & b_2 & a_2 & & & a_N \\ a_1 & b_2 & c_3 & b_3 & a_3 & \mathbf{0} & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \mathbf{0} & & a_{N-4} & b_{N-3} & c_{N-2} & b_{N-2} & a_{N-2} \\ a_{N-1} & & & a_{N-3} & b_{N-2} & c_{N-1} & b_{N-1} \\ b_N & a_N & & & a_{N-2} & b_{N-1} & c_N \end{bmatrix}, \tag{3.26}$$

then $M$ has the unique factorization

$$M = DT'TD, \tag{3.27}$$

where

$$D = \operatorname{diag}\{d_1, d_2, ..., d_N\} \tag{3.28}$$

and

$$T = \begin{bmatrix} 1 & e_1 & f_1 & & \mathbf{0} & g_1 & h_1 \\ & 1 & e_2 & f_2 & & g_2 & h_2 \\ & & \ddots & \ddots & \ddots & | & | \\ & & & 1 & e_{N-4} & f_{N-4} \quad g_{N-4} & h_{N-4} \\ & & & & 1 & e_{N-3} \quad (f_{N-3} + g_{N-3}) & h_{N-3} \\ & \mathbf{0} & & & & 1 \quad (e_{N-2} + g_{N-2}) & (f_{N-2} + h_{N-2}) \\ & & & & & 1 & (e_{N-1} + h_{N-1}) \\ & & & & & & 1 \end{bmatrix}. \tag{3.29}$$

The elements of $D$ and $T$ are then obtained from the following relations:

$$d_1 = c_1^{1/2}, \qquad d_2 = (c_2 - b_1^2/d_1^2)^{1/2}, \qquad e_1 = b_1/d_1 d_2\,;$$

set, for $i = 3, 4, ..., N - 2$,

$$v = a_{i-2}/d_{i-2}\,, \qquad w = (b_{i-1}/d_{i-1}) - e_{i-2}v,$$

then

$$d_i = [c_i - v^2 - \omega^2]^{1/2}, \qquad f_{i-2} = v/d_i\,, \qquad e_{i-1} = w/d_i\,.$$

Let

$$x_1 = a_{N-1}/d_1\,, \qquad x_2 = -e_1 x_1\,,$$
$$v = a_{N-3}/d_{N-3}\,, \qquad w = b_{N-2}/d_{N-2} - e_{N-3}v,$$

and for $i = 3, 4, \ldots, N-2$

$$x_i = -e_{i-1}x_{i-1} - f_{i-2}x_{i-2} .$$

Then

$$d_{N-1} = \left\{ c_{n-1} - \left[ \sum_{i=1}^{N-4} x_i^2 + (x_{N-3} + v)^2 + (x_{N-2} + w)^2 \right] \right\}^{1/2}$$

$$f_{N-3} = v/d_{N-1} , \qquad e_{N-2} = w/d_{N-1} ,$$

and for $i = 1, 2, \ldots, N-2$

$$g_i = x_i/d_{N-1} .$$

Let

$$x_1 = b_N/d_1 , \qquad x_2 = a_N/d_2 - e_1x_1 ,$$
$$v = a_{N-2}/d_{N-2} , \qquad w = b_{N-1}/d_{N-1} - e_{N-2}v,$$

and for $i = 3, 4, \ldots, N-1$

$$x_i = -e_{i-1}x_{i-1} - f_{i-2}x_{i-2} ,$$

and

$$x_{N-1} = x_{N-1} - \sum_{i=1}^{N-2} g_i x_i - g_{N-2}v.$$

Then

$$d_N = \left\{ c_N - \left[ \sum_{i=1}^{N-3} x_i^2 + (x_{N-2} + v)^2 + (x_{N-1} + w)^2 \right] \right\}^{1/2}$$

$$f_{N-2} = v/d_N , \qquad e_{N-1} = w/d_N ,$$

and for $i = 1, 2, \ldots, N-1$

$$h_i = x_i/d_N .$$

Then, to solve the system $M\boldsymbol{\phi} = \mathbf{k}$, rewrite in the form

$$DT'TD\boldsymbol{\phi} = \mathbf{k}, \qquad \text{i.e.,} \qquad T'T(D\boldsymbol{\phi}) = D^{-1}\mathbf{k}.$$

Putting $D\boldsymbol{\phi} = \mathbf{y}$, $D^{-1}\mathbf{k} = \mathbf{q}$, the system becomes $T'T\mathbf{y} = \mathbf{q}$, which can be solved directly for $\mathbf{y}$ in terms of the auxiliary vector $\mathbf{p}$ whose components are given by

$$p_1 = q_1 ,$$
$$p_2 = q_2 - e_1 p_1 ,$$
$$p_i = q_i - e_{i-1}p_{i-1} - f_{i-2}p_{i-2} , \qquad i = 3, 4, \ldots, N-2,$$

$$p_{N-1} = q_{N-1} - e_{N-2}p_{N-2} - f_{N-3}p_{N-3} - \sum_{i=1}^{N-2} g_i p_i ,$$

$$p_N = q_N - e_{N-1}p_{N-1} - f_{N-2}p_{N-2} - \sum_{i=1}^{N-1} h_i p_i .$$

The solution $\mathbf{y}$ is then obtained by back substitution so that

$$\begin{aligned} y_N &= p_N , \\ y_{N-1} &= p_{N-1} - (e_{N-1} + h_{N-1})\, y_N , \\ y_{N-2} &= p_{N-2} - (e_{N-2} + g_{N-2})\, y_{N-1} - (f_{N-2} + h_{N-2})\, y_N , \\ y_i &= p_i - e_i\, y_{i+1} - f_i\, y_{i+2} - g_i\, y_{N-1} - h_i\, y_N , \quad i = N-3, N-4,\ldots, 2, 1. \end{aligned}$$

The actual solution $\boldsymbol{\phi}$ is then obtained from $\boldsymbol{\phi} = D^{-1}\mathbf{y}$ using only one division per component. Thus, an efficient algorithm exists for use in the S2LOR method and can be expressed in a compact form similar to (3.11).

If (1.1) reduces to Laplace's equation, then by using a theory due to Parter [6], Wood [7] shows that the spectral radius of the 2-line block Jacobi matrix satisfies

$$\bar{\mu}_{2L} \simeq 1 - (\pi^2/(m+1)^2), \tag{3.30}$$

so that the corresponding spectral radius of the S2LOR method is

$$\bar{\lambda}_{2L} \simeq 1 - (2\pi/(m+1))\, 2^{1/2}, \tag{3.31}$$

which is again independent of $n$. The corresponding 1-line (i.e., SLOR) spectral radius from (3.8) satisfies

$$\bar{\lambda}_{1L} \simeq 1 - (2\pi/(m+1)), \tag{3.32}$$

showing that here, as in the normal Dirichlet problem, the S2LOR method is asymptotically $2^{1/2}$ times faster than SLOR.

It has been shown, then, that for the chosen periodic problem with the ordering of the points as specified, the convergence rate of the 1- and 2-line successive over-relaxation methods is independent of the number of points in the $x$ direction. Because of the topological equivalence of the regions, similar conclusions hold for the 1- and 2-peripheral methods in a plane region with a hole in it. The problem that does remain, however, is to relate the peripheral techniques to the standard block methods for the usual Dirichlet problem. Using the previous results, one might intuitively proceed as follows.

## 4. SPOR Versus other Block Methods

The problem of solving, say, Laplace's equation in the unit square subject to Dirichlet boundary conditions, might be considered as the limiting case (as the hole size tends to zero) of a unit square with a small hole removed from the center with the function being specified on both boundaries.

If in the usual Dirichlet problem, a square mesh of size $h = 1/n$ is used, then the spectral radii of the point, 1-line and 2-line SOR methods satisfy, respectively,

$$\begin{aligned} \bar{\lambda}^{(D)} &\simeq 1 - (2\pi/n) \\ \bar{\lambda}_{1L}^{(D)} &\simeq 1 - (2(2^{1/2}\pi)/n) \\ \bar{\lambda}_{2L}^{(D)} &\simeq 1 - (4\pi/n). \end{aligned} \tag{4.1}$$

If the region is now considered as being topologically equivalent to the rectangle with the periodicity condition, the number of lines, $m$, in the direction of periodicity is,

$$\begin{aligned} m &= (n-1)/2, \qquad n \text{ odd}, \\ &= (n-2)/2, \qquad n \text{ even}. \end{aligned} \tag{4.2}$$

If we take, for example, $n$ even, substituting into 3.4, 3.32, 3.31, gives,

$$\begin{aligned} \bar{\lambda}^{(P)} &\simeq 1 - (2(2^{1/2}\pi)/n), \\ \bar{\lambda}_{1L}^{(P)} &\simeq 1 - (4\pi/n), \\ \bar{\lambda}_{2L}^{(P)} &\simeq 1 - (4(2^{1/2}\pi)/n), \end{aligned} \tag{4.3}$$

where the superfix $(P)$ denotes the periodic case.

Bearing in mind that the 1-line method for the periodic problem is equivalent to the 1-peripheral (SPOR) for the holed region, a comparison of 4.1 and 4.3 shows that a peripheral point ordering (if consistent) corresponds to the SLOR method, and that SPOR is really nearer to S2POR, i.e., if $R(M)$ denotes the asymptotic rate of convergence of method $M$, then

$$R(\text{Point Peripheral}) \sim R(\text{SLOR}),$$

$$R(\text{SPOR}) \sim 2^{1/2} R(\text{SLOR}) = R(\text{S2LOR}),$$

$$R(\text{S2POR}) \sim 2R(\text{SLOR}).$$

In practice, then, one would certainly expect SPOR to be faster than SLOR if not by the full factor $2^{1/2}$.

## 5. Numerical Results

Problem 1. The solution of Laplace's equation in the unit square subject to the boundary conditions

$$\phi(x, 0) = 0; \qquad \phi(x, 1) = 1,$$

$$\phi(0, y) = \phi(1, y).$$

For the 1-line case (the lines being in the $x$-direction) the experimental value of $\bar{\omega}$ and the value from the theory viz.

$$\bar{\omega} = \frac{2}{1 + \sin(\pi/(m + 1))} \tag{5.1}$$

as obtained from (3.2) and (3.8), agreed extremely well.

For the 2-line case, the approximation (3.30) was used in (3.2) and the values so obtained agreed very well with experiment. The method was, as indicated in Section 3, faster than the 1-line by a factor of $2^{1/2}$. The results are given in Table I.

Problem 2. Laplace's equation in the region $R$ defined to be the unit square with a square hole at the center.

TABLE I

| $m$ | Iterations | $\bar{\omega}$ from experiment | $\bar{\omega}$ from theory |
|---|---|---|---|
| | | SLOR | |
| 5 | 10 | 1.27 | 1.26 |
| 10 | 20 | 1.53 | 1.53 |
| 15 | 30 | 1.66 | 1.66 |
| 20 | 39 | 1.73 | 1.73 |
| 40 | 77 | 1.85 | 1.85 |
| | | S2LOR | |
| 9 | 13 | 1.38 | 1.34 |
| 15 | 21 | 1.55 | 1.54 |
| 19 | 26 | 1.61 | 1.62 |
| 39 | 50 | 1.78 | 1.80 |

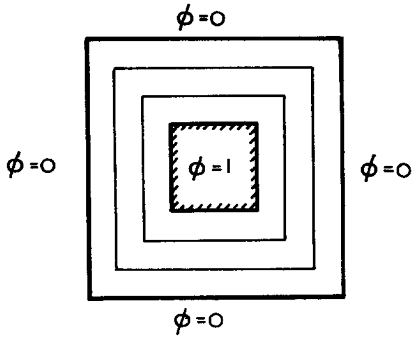


FIGURE 1

The associated boundary conditions were $\phi = 0$ on the outer boundary and $\phi = 1$ on the inner boundary (see Fig. 1).

This problem was considered by Benson and Evans [2], and SPOR compared with standard iterative techniques. It was suggested that the formula (5.1), where $m$ is the number of peripheral blocks, gave quite good agreement with practical results for the optimum value of $\omega$ for the SPOR method. It is interesting to note that (5.1) is exactly the same as that obtained from (3.2) and (3.8) for the 1-line SOR on the rectangle with the periodicity condition. The number of iterations also varied only with $m$ and not with the mesh size, so substantiating the previous

TABLE II

| $m$ | Optimum $\omega$ in practice | $\bar{\omega}$ from 5.1 | No. of iterations |
|---|---|---|---|
| 2 | 1.09 | 1.07 | 7 |
| 4 | 1.29 | 1.26 | 12 |
| 6 | 1.42 | 1.40 | 17 |
| 8 | 1.52 | 1.49 | 21 |
| 10 | 1.59 | 1.56 | 26 |
| 12 | 1.64 | 1.61 | 31 |
| 14 | 1.68 | 1.66 | 37 |
| 16 | 1.72 | 1.69 | 42 |
| 18 | 1.76 | 1.72 | 47 |

theory. Selected results are given in Table II for a mesh size $h = 1/40$ and the size of the hole varying to accomodate the different number of peripherals.

PROBLEM 3. Laplace's equation in the ring $a \leqslant r \leqslant 1$, i.e.,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0,$$

subject to the boundary conditions

$$\phi = 0, \text{ on } r = a; \qquad \phi = 1, \text{ on } r = 1.$$

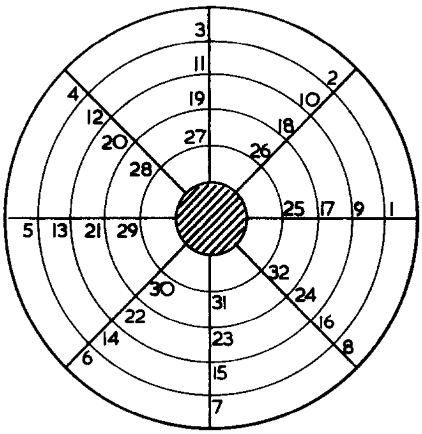


FIGURE 2

This problem was also considered by Benson and Evans [2]. Ordering the mesh points in the $(r, \theta)$ plane along successive circumferences makes the problem equivalent to the one treated in Sections 1–3 (see Fig. 2). It could be expected then, that the rate of convergence would depend only on $m$, and that (5.1) would give a reasonable approximation to the optimum value of $\omega$. That this is in fact so, can be seen from the results in Table III.

A two-peripheral grouping of the mesh points would also be expected to exhibit the same type of behavior and to be faster than SPOR by a factor of $2^{1/2}$. This is in fact so, and the results are given in Table IV.

The ordering is shown in Fig. 3.

TABLE III

| $M^a$ | $N^b$ | $m$ | Iterations | $\bar{\omega}$ in practice | $\bar{\omega}$ from (5.1) |
|---|---|---|---|---|---|
| 10 | 40 | 3 | 7 | 1.175 | 1.17 |
| | | 4 | 9 | 1.26 | 1.26 |
| | | 5 | 11 | 1.33 | 1.33 |
| | | 6 | 13 | 1.395 | 1.39 |
| | | 7 | 15 | 1.455 | 1.45 |
| | | 8 | 18 | 1.51 | 1.49 |
| 20 | 80 | 3 | 7 | 1.165 | 1.17 |
| | | 4 | 9 | 1.245 | 1.26 |
| | | 5 | 11 | 1.31 | 1.33 |
| | | 6 | 13 | 1.37 | 1.39 |
| | | 7 | 15 | 1.42 | 1.45 |
| | | 8 | 17 | 1.46 | 1.49 |
| | | 10 | 21 | 1.54 | 1.57 |
| | | 12 | 25 | 1.60 | 1.62 |
| | | 14 | 29 | 1.65 | 1.67 |
| | | 16 | 33 | 1.69 | 1.70 |
| | | 18 | 38 | 1.73 | 1.73 |

[a] $M = \delta r^{-1}$.
[b] $N = 2\pi/\delta\theta$.

TABLE IV

| $M$ | $N$ | $m$ | Iterations | $\bar{\omega}$ in practice |
|---|---|---|---|---|
| 10 | 40 | 4 | 8 | 1.17 |
| | | 6 | 10 | 1.28 |
| | | 8 | 14 | 1.39 |
| 20 | 80 | 4 | 8 | 1.175 |
| | | 6 | 11 | 1.285 |
| | | 8 | 13 | 1.38 |
| | | 10 | 16 | 1.45 |
| | | 12 | 19 | 1.51 |
| | | 14 | 22 | 1.56 |
| | | 16 | 26 | 1.61 |
| | | 18 | 28 | 1.65 |

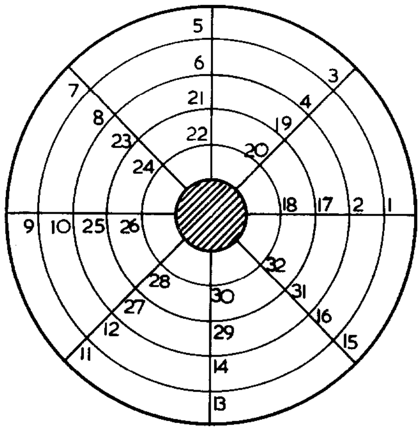


FIGURE 3

## 6. Conclusions

Algorithms have been derived for solving the sets of equations arising from the 1-line and 2-line successive over-relaxation methods used for solving the Dirichlet/Periodic problem considered theoretically by Wood [7]. The numerical results obtained substantiate the theory.

It has also been shown that the theory given by Wood [7] can be used for a peripheral ordering of the points in a certain class of problems, i.e., those involving a plane region with a hole for which the peripheral techniques seem ideally suited. Again, numerical evidence of the validity of the theory has been given. Finally, an attempt has been made to link the peripheral and line techniques in the case of the normal Dirichlet problem. It is shown that the peripheral grouping is more closely analogous to the 2-line grouping; it can certainly be expected to be faster than the 1-line successive over-relaxation method, probably because it more rapidly utilizes the boundary mesh points to compute the interior mesh points.

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